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# Siegel Zeros and the Goldbach Problem

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It is generally known that under the generalized Riemann hypothesis one could establish the Goldbach conjecture by the circle method provided one could obtain a certain estimate for the integral of the representation function over the minor arcs. Here it is first shown that the generalized Riemann hypothesis in the above statement can be weakened to the assumption that Siegel zeros do not exist. The case when Siegel zeros do exist is then considered.

## 1. INTRODUCTION

Let $N \geqslant N_0$ and $\log^{15} N \leqslant P \leqslant N^{\varepsilon}$ for $\varepsilon > 0$. Let $x_0 = x_0(N) = P/N$.

When $0 < h \leqslant q \leqslant P$ and $(h, q) = 1$, let $M(q, h)$ denote the closed interval $[h/q - x_0, h/q + x_0]$, a so-called major arc.

It is easily shown, for any choice of $P$, that all the $M(q, h)$ are disjoint and contained in the closed interval $[x_0, 1 + x_0]$.

For each $N$ let $m(N)$ be those points in $[x_0, 1 + x_0]$ which are not in any closed neighborhood (major arc) of radius $x_0$ about any rational number $h/q$, where $(h, q) = 1$ and $q \leqslant P$.

For each $N$ let $m^*(N)$ be those points in $[x_0, 1 + x_0]$ which are not in any closed neighborhood (major arc) of radius $x_0$ about any rational number $h/q$, where $(h, q) = 1$, $(q, N) = 1$, and $q \leqslant P$.

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Let $p$ denote a prime, and let $E(\alpha)=\exp(2\pi i\alpha)$.
Let

$$f(x,N)=\sum_{p\leqslant N} E(px) \qquad \text{and} \qquad r(N)=\sum_{\substack{p_1,p_2\\ p_1+p_2=N}} 1.$$

Let

$$J(N)=\sum_{\substack{N_1,N_2\geqslant 2\\ N_1+N_2=N}} (\log N_1 \log N_2)^{-1}.$$

Let

$$S(N)=(1+(-1)^N)\left\{\prod_{p\geqslant 3}\left(1-\frac{1}{(p-1)^2}\right)\right\}\prod_{\substack{p\geqslant 3\\ p\mid N}}\left(\frac{p-1}{p-2}\right).$$

Let

$$C_q(N)=\sum_{\substack{0<h\leqslant q\\ (h,q)=1}} E\left(-\frac{Nh}{q}\right) \qquad \text{and} \qquad R(N)=\sum_{q\leqslant P}\frac{u^2(q)}{\phi^2(q)}C_q(N).$$

In [10] we established the following two theorems:

THEOREM 1.1. *Under the generalized Riemann hypothesis with $P=N^\varepsilon$, if $\int_{m(N)} f^2(x,N)\,E(-Nx)\,dx=o(N\log^{-2}N)$, then $r(N)>0$ for all even $N\geqslant N_0$.*

THEOREM 1.2. *Let $P=N^{-1}\log^{15}N$. If $\int_{m^*(N)} f^2(x,N)\,E(-Nx)\,dx=o(N\log^{-2}N)$, then $r(N)>0$ for all even $N\geqslant N_0$.*

In Section 3 of this paper we establish

THEOREM 1.3. *Under the assumption that Siegel zeros do not exist with $P=\exp(c\log^{1/2}N)$ if $\int_{m(N)} f^2(x,N)\,E(-Nx)\,dx=o(N\log^{-2}N)$, then $r(N)>0$ for all even $N\geqslant N_0$.*

In Section 4 of this paper we establish

THEOREM 1.4. *Let $P=\exp(c\log^{1/2}N)$. If $\int_{m(N)} f^2(x,N)\,E(-Nx)\,dx=o(NP^{-1/32}\log^{-2}N)$, then $r(N)>0$ for all even $N\geqslant N_0$.*

In Section 6 of this paper we show that a particular natural approach for eliminating the condition $(q,N)=1$ in Theorem 1.2 will not work.

## 2. Preliminary Lemmas

Lemma 2.1. *We have* $\sum_{p|n}(1/p^{5/8}) = O(\log^{3/8} n)$.

*Proof.* Fix $y > 0$. Then

$$\sum_{p|n} \frac{1}{p^{5/8}} = \sum_{\substack{p|n\\ p<y}} \frac{1}{p^{5/8}} + \sum_{\substack{p|n\\ p\geqslant y}} \frac{1}{p^{5/8}} = A(n) + B(n).$$

It is trivial that $A(n) = O(y^{3/8})$ and

$$B(n) \leqslant \frac{1}{y^{5/8}\log y} \sum_{p|n} \log p \leqslant C\frac{\log n}{y^{5/8}\log y}.$$

Let $y = \log n$, and the proof follows.

Lemma 2.2. *We have* $\sum_{m|n}(1/m^{5/8}) = O(\exp(c\log^{3/8} n))$.

*Proof.* Let $A(n) = \sum_{m|n}(1/m^{5/8})$. By the Euler product formula we have

$$A(n) \leqslant \prod_{p/n}\left(1 + \sum_{j=1}^{\infty} \frac{1}{(p^j)^{5/8}}\right) \leqslant \prod_{p|n}(1 + 100p^{-5/8}),$$

so that $\log A(n) \leqslant \sum_{p|n}\log(1 + 100p^{-5/8}) \leqslant C\sum_{p|n} p^{-5/8}$, and the proof follows from Lemma 2.1.

Lemma 2.3. *We have* $\sum_{m|n, m\leqslant y} 1 = O(y^{5/8}\exp(c\log^{3/8} n))$.

*Proof.* This is immediate by Lemma 2.2.

Lemma 2.4. *We have* $\phi^{-1}(m) \ll m^{-1}(\log\log(m+3))$ *if* $m \geqslant 1$.

*Proof.* This is immediate by [6, Theorem 328].

Lemma 2.5. *We have*

$$\sum_{P<q} \frac{\mu^2(q)}{\phi^2(q)} C_q(N) \ll P^{-3/8}\exp(\mathrm{c}\log^{3/8} N).$$

*Proof.* It is shown in [12, p. 27] that

$$\sum_{P<q} \frac{\mu^2(q)}{\phi^2(q)} C_q(N) = \sum_{k|N} \frac{\mu^2(k)}{\phi^2(k)} \sum_{\substack{q>P/k\\ (q,N)=1}} \frac{\mu(q)}{\phi^2(q)} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{\substack{k \mid N \\ k \leqslant P}} \frac{\mu^2(k)}{\phi(k)} \sum_{\substack{q > P/k \\ (q,N)=1}} \frac{\mu(q)}{\phi^2(q)} \quad \text{and} \quad \Sigma_2 = \sum_{\substack{k \mid N \\ k > P}} \frac{\mu^2(k)}{\phi(k)} \sum_{\substack{q \geqslant 1 \\ (q,N)=1}} \frac{\mu(q)}{\phi^2(q)}$$

By Lemma 2.4 we have

$$\Sigma_1 \ll P^{-1}(\log\log(P+3))^3 \sum_{\substack{k \mid N \\ k \leqslant P}} 1,$$

and by Lemma 2.3 this yields

$$\Sigma_1 \ll P^{-3/8}(\log\log(P+3))^3 \exp(c \log^{3/8} N).$$

But $(\log\log(P+3))^3 \leqslant \exp(c_1 \log^{3/8} P) \leqslant \exp(c_1 \log^{3/8} N)$, so that

$$\Sigma_1 \ll P^{-3/8} \exp(c' \log^{3/8} N).$$

$$\begin{aligned}
\Sigma_2 &\ll \sum_{\substack{k \mid N \\ k > P}} \frac{1}{\phi(k)} \sum_{\substack{q \geqslant 1 \\ (q,N)=1}} \frac{1}{\phi^2(q)} \\
&\ll \sum_{\substack{k \mid N \\ k > P}} \frac{1}{\phi(k)} = \sum_{\substack{k \mid N \\ k > P}} \frac{k}{k\phi(k)} \ll \log\log(k+3) \sum_{\substack{k \mid N \\ k > P}} \frac{1}{k} \\
&\ll \log\log(N+3) \sum_{\substack{k > P \\ k \mid N}} \frac{1}{k}\left(\frac{k}{P}\right)^{3/8} \ll P^{-3/8} \log\log(N+3) \sum_{k \mid N} \frac{1}{k^{5/8}},
\end{aligned}$$

so that by Lemma 2.2

$$\Sigma_2 \ll P^{-3/8} \log\log(N+3) \exp(c \log^{3/8} N) \ll P^{-3/8} \exp(c'' \log^{3/8} N).$$

LEMMA 2.6. *We have* $S(N) = \sum_{q=1}^{\infty} (\mu^2(q)/\phi^2(q))\, C_q(N)$.

*Proof.* This is [4, Lemma 12] with $r = 2$.

LEMMA 2.7. *We have* $S(N) \gg 1$.

*Proof.* This is established in [10, p. 18].

LEMMA 2.8. *Let* $P = \exp(c(\log N)^{1/2})$ *for any* $c > 0$. *Then*

$$S(N) - R(N) = o(1)$$

*Proof.* Immediate by Lemmas 2.5 and 2.6.

LEMMA 2.9. *We have*

$$\sum_{n \leqslant x} \frac{1}{\phi(n)} = \frac{\xi(2)\,\xi(3)}{\xi(6)} \log x + A + O\left(\frac{\log x}{x}\right).$$

*Proof.* This is established in [7, p. 38].

Let

$$ls_x(X) = \sum_{2 \leqslant m \leqslant X} m^{x-1}(\log m)^{-1}, \tag{2.1}$$

$$ls\, X = ls_1(X). \tag{2.2}$$

LEMMA 2.10. *There are positive numbers $C_{11}$, $C_{12}$, and $C_{13}$ such that, for every sufficiently large number $N$,*

(i) *For every $q$, $h$ such that $q \leqslant \exp((\log N)^{1/2})$ and $(q, h) = 1$ we have whenever $N^{3/4} < X \leqslant N$,*

$$|\pi(X, q, h) - (ls\, X/\phi(q))| < C_{11} X \exp(-C_{12}(\log X)^{1/2}),$$

*or*

(ii) *there is just one pair $r, \beta$ such that for every $q, h$ with $q \leqslant \exp((\log N)^{1/2})$ and $(q, h) = 1$, and every $X$ with $N^{3/4} < X \leqslant N$, we have*

$$|\pi(X, q, h) - (ls\, X/\phi(q))| < C_{11} X \exp(-C_{12}(\log X)^{1/2}) \qquad (r \nmid q)$$

*and*

$$\left| \pi(X, q, h) - \frac{ls\, X}{\phi(q)} + \frac{\chi(h)}{\phi(q)} ls_\beta(X) \right| < C_{11} X \exp(-C_{12}(\log X)^{1/2}) \qquad (r \mid q),$$

*where $\chi$ is the real nonprincipal character modulo $q$ induced in each case by the same real nonprincipal primitive character modulo $r$. Moreover,*

$$\tfrac{1}{2} \leqslant \beta < 1 - C_{13} r^{-1/8} \tag{2.3}$$

*and*

$$r > (\log N)^3. \tag{2.4}$$

*Proof.* This is established in [12, p. 26].

## 3. A Proof of Theorem 1.3

Let $q \leqslant P$, $|y| \leqslant x_0$, $(h, q) = 1$, and $N \geqslant N_0$. Then

$$\left| f\left(\frac{h}{q} + y, N\right) - \frac{\mu(q)}{\phi(q)} g(y, N) \right| \leqslant \left| f\left(\frac{h}{q}, N\right) - \frac{\mu(q)}{\phi(q)} g(0, N) \right| + 2\pi x_0 \int_0^N \left| f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0, v) \right| dv,$$

where $g(x, v) = \sum_{2 \leqslant m \leqslant v} (E(mx)/\log m)$ if $v \geqslant 2$ and $g(x, v) = 0$ if $v < 2$. Hence

$$\begin{aligned} &\left| f\left(\frac{h}{q} + y, N\right) - \frac{\mu(q)}{\phi(q)} g(y, N) \right| \\ &\quad \leqslant \left| f\left(\frac{h}{q}, N\right) - \frac{\mu(q)}{\phi(q)} g(0, N) \right| \\ &\qquad + 2\pi x_0 \int_0^N q + \left| \sum_{\substack{0 < l \leqslant q \\ (l, q) = 1}} E\left(\frac{lh}{q}\right) \left\{ \pi([v]; q, l) - \frac{ls[v]}{\phi(q)} \right\} \right| dv. \end{aligned}$$

Using the fact that if $|a| \leqslant N$ and $|b| \leqslant N$, then $|a^2 - b^2| \leqslant 2N |a - b|$, it follows that

$$\left| f^2 \left(\frac{h}{q} + y, N\right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y, N) \right| \leqslant F(h, q, N),$$

where

$$\begin{aligned} F(h, q, N) = 2N &\left| f\left(\frac{h}{q}, N\right) - \frac{\mu(q)}{\phi(q)} g(0, N) \right| \\ &+ 4\pi x_0 N \int_0^N q + \left| \sum_{\substack{0 < l \leqslant q \\ (l, q) = 1}} E\left(\frac{lh}{q}\right) \left\{ \pi([v]; q, l) - \frac{ls[v]}{\phi(q)} \right\} \right| dv. \end{aligned}$$

By a change of variable $y = x - (h/q)$ it follows that

$$T(h, q) = E\left(-\frac{Nh}{q}\right) \int_{-x_0}^{x_0} f^2 \left(\frac{h}{q} + y, N\right) E(-Ny)\, dy. \tag{3.1}$$

But

$$\left| E\left(-\frac{Nh}{q}\right)\int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y, N\right)E(-Ny)\,dy - \frac{\mu^2(q)}{\phi^2(q)}E\left(-\frac{Nh}{q}\right)\int_{-x_0}^{x_0} g^2(y, N)\,E(-Ny)\,dy \right|$$
$$\leqslant \int_{-x_0}^{x_0}\left| f^2\left(\frac{h}{q}+y, N\right) - \frac{\mu^2(q)}{\phi^2(q)}g^2(y, N)\right| dy$$
$$\leqslant \int_{-x_0}^{x_0} F(h, q, N)\,dy = 2x_0F(h, q, N).$$

Let $T_1(N) = \int_{-x_0}^{x_0} g^2(y, N)\,E(-Ny)\,dy$, so that by (3.1) it follows that if $q \leqslant P$, $(h, q) = 1$, and $N \geqslant N_0$, then

$$\left| T(h, q) - \frac{\mu^2(q)}{\phi^2(q)}T_1(N)\,E\left(-\frac{Nh}{q}\right)\right| \leqslant 2x_0F(h, q, N). \tag{3.2}$$

Let $T(N) = \sum_{m_1, m_2} \log^{-1} m_1 \log^{-1} m_2$ with the conditions of summation $m_1 \geqslant 2$, $m_2 \geqslant 2$, and $m_1 + m_2 = N$. It is easy to see that

$$\left| E\left(-\frac{Nh}{q}\right)\right| \left|\frac{\mu^2(q)}{\phi^2(q)}\right| |T(N) - T_1(N)| \leqslant \frac{2}{x_0\phi^2(q)},$$

and combining this fact with (3.2) it follows that

$$\left| T(h, q) - \frac{\mu^2(q)}{\phi^2(q)}T(N)\,E\left(-\frac{Nh}{q}\right)\right| \leqslant 2x_0F(h, q, N) + \frac{2}{x_0\phi^2(q)}. \tag{3.3}$$

Adding (3.3) $\phi(q)$ times for some fixed $q \leqslant P$ it follows that

$$\left| \sum_{\substack{0<h\leqslant q\\(h,q)=1}} T(h, q) - \frac{\mu^2(q)}{\phi^2(q)}T(N)\sum_{\substack{0<h\leqslant q\\(h,q)=1}} E\left(-\frac{Nh}{q}\right)\right|$$
$$\leqslant \sum_{\substack{0<h\leqslant q\\(h,q)=1}} 2x_0F(h, q, N) + \frac{2}{x_0\phi(q)}.$$

Now summing over all $q \leqslant P$ it follows that

$$\left| \sum_{q\leqslant P}\sum_{\substack{0<h<q\\(h,q)=1}} T(h, q) - T(N)\sum_{q\leqslant P}\frac{\mu^2(q)}{\phi^2(q)}C_q(N)\right|$$
$$\leqslant \sum_{q\leqslant P}\sum_{\substack{0<h\leqslant q\\(h,q)=1}} 2x_0F(h, q, N) + \sum_{q\leqslant P}\frac{2}{x_0\phi(q)}. \tag{3.4}$$

LEMMA 3.1. *We have* $N \log^{-2} N \ll T(N)$.

*Proof.* This is established in [10, p. 15].

LEMMA 3.2. *We have* $\sum_{q \leqslant P} (2/x_0 \phi(q)) = o(N \log^{-2} N)$.

*Proof.* This is a straightforward consequence of Lemma 2.9.

LEMMA 3.3. *We have* $F(h, q, N) \ll (PN)^2 \exp(-c \log N)^{1/2})$.

*Proof.* We have $F(h, q, N) \leqslant A_1 + A_2 + A_3$, where

$$A_1 = 2N \left| f\left(\frac{h}{q}, N\right) - \frac{u(q)}{\phi(q)} g(0, N) \right|, \qquad A_2 = 4\pi x_0 N^2 q,$$

$$A_3 = 4\pi x_0 N \int_0^N \sum_{\substack{0 < l \leqslant q \\ (l,q)=1}} \left| \pi([v]; q, l) - \frac{ls[v]}{\phi(q)} \right| dv.$$

We estimate $A_1$:

$$\begin{aligned} &\left| f\left(\frac{h}{q}, N\right) - \frac{u(q)}{\phi(q)} g(0, N) \right| \\ &\quad \leqslant q + \sum_{\substack{0 < l < q \\ (l,q)=1}} \left| \pi([N]; q, l) - \frac{ls[N]}{\phi(q)} \right| \\ &\quad \leqslant q + \sum_{0 < l \leqslant q} C_{11} N \exp(-C_{12} (\log N)^{1/2}) \qquad \text{(by Lemma 2.10)} \\ &\quad \leqslant C_{13} PN \exp(-C_{12} (\log N)^{1/2}), \end{aligned}$$

so that

$$A_1 \leqslant C_{14} PN^2 \exp(-C_{12} (\log N)^{1/2}).$$

We estimate $A_3$: let $A_3 = A_3' + A_3''$, where

$$A_3' = 4\pi x_0 N \int_0^{N^{7/8}} \sum_{\substack{0 < l \leqslant q \\ (l,q)=1}} \left| \pi([v]; q, l) - \frac{ls[v]}{\phi(q)} \right| dv$$

and

$$A_3'' = 4\pi x_0 N \int_{N^{7/8}}^N \sum_{\substack{0 < l \leqslant q \\ (l,q)=1}} \left| \pi([v]; q, l) - \frac{ls[v]}{\phi(q)} \right| dv.$$

Using the trivial estimates

$$|\pi([v]; q, l)| \leqslant v \qquad \text{and} \qquad |ls[v]/\phi(q)| \leqslant v,$$

it is easy to see that $A_3' \leqslant C_{15} P^2 N^{7/4}$. By Lemma 2.10 it follows easily that $A_3'' \leqslant C_{16} P^2 N^2 \exp(-C_{17}(\log N)^{1/2})$. Clearly, $A_2 \leqslant C_{17} P^2 N$, so the proof is completed.

LEMMA 3.4. *Let $P = \exp(c_0(\log N)^{1/2})$. It is possible to choose a positive $c_0 < 1$ small enough such that*

$$\sum_{q \leqslant P} \sum_{\substack{0 < h \leqslant q \\ (h,q)=1}} 2x_0 F(h, q, N) = o(N \log^{-2} N).$$

*Proof.* This follows immediately from Lemma 3.3.

The proof of Theorem 1.3 follows from (3.4) and Lemmas 2.7, 2.8, 3.1, 3.2, and 3.4.

## 4. A PROOF OF THEOREM 1.4

Let $C_{14} = \min(1, (\sqrt{3}/2)\, C_{12}, C^*)$, where $0 < C^* \leqslant c_0$ is to be chosen later. Let $P_1 = \exp(\frac{1}{2} C_{14}(\log N)^{1/2})$ and let $P_2 = P_1^{1/4}$. Fix $N \geqslant N_0$.

*Case* 1. $r > P_2$. Let $P = P_2$, so that none of the $q$'s are divided by $r$. Then the proof is immediate by the same argument used to establish Theorem 1.3.

*Case* 2. $r \leqslant P_2$. Let $P = P_1$. First, consider all $q$ such that $r \nmid q$. By the same argument used in the proof of Theorem 1.3 it follows that

$$\begin{aligned} &\left| \sum_{\substack{q \leqslant P \\ r \nmid q}} \sum_{\substack{0 < h \leqslant q \\ (h,q)=1}} T(h, q) - T(N) \sum_{\substack{q \leqslant P \\ r \nmid q}} \frac{u^2(q)}{\phi^2(q)} C_q(N) \right| \\ &\qquad \leqslant \sum_{\substack{q \leqslant P \\ r \nmid q}} \sum_{\substack{0 < h \leqslant q \\ (h,q)=1}} 2x_0 F(h, q, N) + \sum_{\substack{q \leqslant P \\ r \nmid q}} \frac{2}{x_0 \phi(q)}. \end{aligned} \tag{4.1}$$

Now consider all $q$ such that $r \mid q$. It is assumed that $|y| \leqslant x_0$, $(h, q) = 1$, and $0 \leqslant v \leqslant N$. Using the fact that $ls[v] = g(0, v)$, it follows that for any $F_0(h, q, v)$

$$\left|f\left(\frac{h}{q}, v\right) - \frac{u(q)}{\phi(q)} g(0, v) + F_0(h, q, v)\right|$$
$$\leqslant \left|f\left(\frac{h}{q}, v\right) - \sum_{\substack{p \leqslant v \\ p \nmid q}} E\left(\frac{ph}{q}\right)\right|$$
$$+ \left|\sum_{\substack{p \leqslant v \\ p \nmid q}} E\left(\frac{ph}{q}\right) - \frac{u(q)}{\phi(q)} ls[v] + F_0(h, q, v)\right|.$$

Let

$$F_0(h, q, v) = \sum_{\substack{0 < l < q \\ (l, q) = 1}} E\left(\frac{lh}{q}\right) \frac{\chi(l)}{\phi(q)} ls_\beta(v).$$

Using the facts that

$$\sum_{\substack{p \leqslant v \\ p \nmid q}} E\left(\frac{ph}{q}\right) = \sum_{\substack{0 < l \leqslant q \\ (l, q) = 1}} \left(E\left(\frac{lh}{q}\right) \pi([v]; q, l)\right) \quad \text{and} \quad \sum_{\substack{0 < l \leqslant q \\ (l, q) = 1}} E\left(\frac{lh}{q}\right) = \mu(q)$$

it follows that for $0 < v \leqslant N$

$$\left|f\left(\frac{h}{q}, v\right) - \frac{u(q)}{\phi(q)} g(0, v) + F_0(h, q, v)\right|$$
$$\leqslant q + \left|\sum_{\substack{0 < l \leqslant q \\ (l, q) = 1}} E\left(\frac{lh}{q}\right) \left\{\pi([v]; q, l) - \frac{ls[v]}{\phi(q)} + \frac{\chi(l)}{\phi(q)} ls_\beta(v)\right\}\right|$$

Let $F^*(h, q, N) = E(Ny) F_0(h, q, N) - 2\pi i y \int_0^N E(vy) F_0(h, q, v)\, dv$. It is easy to see by straightforward calculation that

$$f\left(\frac{h}{q} + y, N\right) - \frac{u(q)}{\phi(q)} g(y, N) + F^*(h, q, N)$$
$$= E(Ny) f\left(\frac{h}{q}, N\right) - \frac{u(q)}{\phi(q)} E(Ny)\, g(0, N) + E(Ny)\, F_0(h, q, N)$$
$$- 2\pi i y \int_0^N E(vy) \left\{f\left(\frac{h}{q}, v\right) - \frac{u(q)}{\phi(q)} g(0, v) + F_0(h, q, v)\right\} dv.$$

so that using the fact that if $|a| \leqslant N$ and $|b| \leqslant N$, then $|a^2 - b^2| \leqslant 2N$ it follows that

$$\left|f^2\left(\frac{h}{q} + y, N\right) - \left[\frac{u(q)}{\phi(q)} g(y, N) - F^*(h, q, N)\right]^2\right| \leqslant F_1(h, q, N),$$

where

$$F_1(h,q,N)=2N\left|f\left(\frac{h}{q},N\right)-\frac{u(q)}{\phi(q)}g(0,N)+F_0(h,q,N)\right|$$

$$+4\pi x_0N\int_0^N\left(q+\left|\sum_{\substack{0<l\leqslant q\\(l,q)=1}}E\left(\frac{lh}{q}\right)\right\}\pi(|v|;q,l)\right.$$

$$\left.-\frac{ls[v]}{\phi(q)}+\frac{\chi(l)}{\phi(q)}ls_\beta(v)\Big\}\Big|\right)dv.$$

Let

$$T_1(N)=T_1(h,q,N)=\int_{-x_0}^{x_0}g^2(y,N)E(-Ny)\,dy,$$

$$T_2(N)=T_2(h,q,N)=\int_{-x_0}^{x_0}F^*(h,q,N)g(y,N)E(-Ny)\,dy,$$

and

$$T_3(N)=T_3(h,q,N)=\int_{-x_0}^{x_0}F^{*2}(h,q,N)E(-Ny)\,dy.$$

Let

$$A(N)=\sum_{\substack{q\leqslant P\\r|q}}\sum_{\substack{0<h\leqslant q\\(h,q)=1}}T_3(N)E\left(-\frac{Nh}{q}\right)-2\sum_{\substack{q\leqslant P\\r|q}}\sum_{\substack{0<h\leqslant q\\(h,q)=1}}\frac{\mu(q)}{\phi(q)}T_2(N)E\left(-\frac{Nh}{q}\right).$$

By straightforward calculation it follows from the above, (4.1), and Lemma 2.5 that

$$\left|\sum_{q\leqslant P}\sum_{\substack{0<h<q\\(h,q)=1}}T(h,q)-[T(N)S(N)+A(N)]\right|$$

$$\leqslant\sum_{\substack{q\leqslant P\\r\nmid q}}\sum_{\substack{0<h\leqslant q\\(h,q)=1}}2x_0F(h,q,N)+\sum_{\substack{q\leqslant P\\r|q}}\sum_{\substack{0<h\leqslant q\\(h,q)=1}}2x_0F_1(h,q,N)$$

$$+\sum_{q\leqslant P}\frac{2}{x_0\phi(q)}+O(cNP^{-3/8}\exp(c\log^{3/8}N).$$

It is not difficult to see using the same methods of proof used to establish Theorem 1.3 that it is possible to choose $C^*$ small enough such that each term on the right-hand side of the inequality is $o(NP^{-1/32}\log^{-2}N)$. Consequently, Theorem 1.4 follows from

LEMMA 4.1. *We have* $T(N)\,S(N)+A(N)\gg NP^{-1/32}\log^{-2}N$ *if* $N$ *even.*

## 5. A PROOF OF LEMMA 4.1

In this section the notation employed in [12] is combined with the notation of the previous sections of this paper. Also, here we use several lemmas from [12]. In all such lemmas $x_0/q$ can be replaced by $x_0$ as far as our application of the lemmas is concerned.

Let

$$G(h,q)=\sum_{\substack{0<l\leqslant q\\(l,q)=1}} E\left(\frac{lh}{q}\right)\chi(l).$$

Then

$$\chi(h)\,G(h,q)=\sum_{\substack{0<l\leqslant q\\(l,q)=1}} E\left(\frac{lh}{q}\right)\chi(l)\,\chi(h)=\sum_{\substack{0<l\leqslant q\\(l,q)=1}} E\left(\frac{lh}{q}\right)\chi(lh)$$
$$=\sum_{\substack{0<l\leqslant q\\(l,q)=1}} E\left(\frac{l}{q}\right)\chi(l),$$

since $lh$ and $l$ run through the same reduced residue classes, so that

$$G(h,q)=\frac{1}{\chi(h)}\sum_{\substack{0<l\leqslant q\\(l,q)=1}} E\left(\frac{l}{q}\right)\chi(l)=\chi(h)\sum_{0<l\leqslant q} E\left(\frac{l}{q}\right)\chi(l).$$

Hence, by [12, Definition 8.3]

$$G(h,q)=\chi(h)\,\tau(\chi).$$

Hence by [12, Definition 8.4] we have

$$F_0(h,q,v)=ls_\beta(v)\,\lambda(q,h),$$

so that by [12, Eq. (5.6)] it follows that

$$F_0(h,q,v)=g_\beta(v,0)\,\lambda(q,h).$$

LEMMA 5.1. *For any real numbers $x_1$ and $x_2$*

$$g_\beta(v, x_1 + x_2) = E(vx_2)\, g_\beta(v_1 x_1) - 2\pi i x_2 \int_0^v E(ux_2)\, g_\beta(u_1, x_1)\, du.$$

*Proof.* Confer [3, p. 63].

By letting $x_1 = 0$, $x_2 = y$, and $v = N$ in Lemma 5.1 we have

$$g_\beta(N, y) = E(Ny)\, g_\beta(N, 0) - 2\pi i y \int_0^N E(uy)\, g_\beta(u, 0)\, du$$

and

$$\lambda(q, h)\, g_\beta(N, y) = E(Ny)\, \lambda(q, h)\, g_\beta(N, 0) - 2\pi i y \int_0^N E(uy)\, g_\beta(u, 0)\, \lambda(q, h)\, du.$$

Hence $F^*(h, q, N) = \lambda(q, h)\, g_\beta(N, y)$, so that

$$T_2(N) = \int_{-x_0}^{x_0} \lambda(q, h)\, g_\beta(N, y)\, g(y, N)\, E(-Ny)\, dy$$

and

$$T_3(N) = \int_{-x_0}^{x_0} \lambda^2(q, h)\, g_\beta^2(N, y)\, E(-Ny)\, dy.$$

By a change of variable, $x = y + (h/q)$, it follows that

$$T_3(N) = \int_{(h/q) - x_0}^{(h/q) + x_0} \lambda^2(q, h)\, g_\beta^2 \left(N, x - \frac{h}{q}\right) E\left(-N\left(x - \frac{h}{q}\right)\right) dx,$$

so that by [12, Eq. (8.5)] it follows that

$$T_3(N) = \int_{(h/q) - x_0}^{(h/q) + x_0} v_\beta^{*2}(N, x, q, h)\, E\left(-N\left(x - \frac{h}{q}\right)\right) dx.$$

Hence

$$\begin{aligned}
&\sum_{\substack{q \leqslant P \\ r \mid q}} \sum_{\substack{0 < h \leqslant q \\ (h, q) = 1}} T_3(N)\, E\left(-\frac{Nh}{q}\right) \\
&\qquad = \sum_{\substack{q \leqslant P \\ r \mid q}} \sum_{\substack{0 < h \leqslant q \\ (h, q) = 1}} \int_{(h/q) - x_0}^{(h/q) + x_0} v_\beta^{*2}(N, x, q, h)\, E(-Nx)\, dx.
\end{aligned}$$

By a change of variable, $x = y + (h/q)$, it follows that

$$T_2(N) = \int_{(h/q)-x_0}^{(h/q)+x_0} \lambda(q, h)\, g_\beta\left(N, x - \frac{h}{q}\right) g\left(x - \frac{h}{q}, N\right) E\left(-N\left(x - \frac{h}{q}\right)\right) dx,$$

so that by [12, Eq. (8.5)] it follows that

$$T_2(N) = \int_{(h/q)-x_0}^{(h/q)+x_0} v_\beta^*(N, x, q, h)\, g\left(x - \frac{h}{q}, N\right) E\left(-N\left(x - \frac{h}{q}\right)\right) dx.$$

Hence by [12, Definition 5.7]

$$2 \sum_{\substack{q \leqslant P \\ r|q}} \sum_{\substack{0<h\leqslant q \\ (h,q)=1}} \frac{\mu(q)}{\phi(q)} T_2(N)\, E\left(-\frac{Nh}{q}\right)$$
$$= \sum_{\substack{q \leqslant P \\ r|q}} \sum_{\substack{0<h\leqslant q \\ (h,q)=1}} \int_{(h/q)-x_0}^{(h/q)+x_0} 2v_\beta^*(N, x, q, h)\, v^*(N, x, q, h)\, E(-Nx)\, dx.$$

Hence

$$A(N) = \sum_{\substack{q \leqslant P \\ r|q}} \sum_{\substack{0<h\leqslant q \\ (h,q)=1}} \int_{(h/q)-x_0}^{(h/q)+x_0} [v_\beta^*(N, x, q, h)^2 - 2v_\beta^*(N, x, q, h)\, v^*(N, x, q, h)]\, E(-Nx)\, dx.$$

Let

$$W(N, x) = v_\beta^*(N, x, q, h)^2 - 2v_\beta^*(N, x, q, h)\, v^*(N, x, q, h), \quad \text{if } x \in M(q, h),$$
$$= 0, \qquad \text{otherwise.}$$

Let $D(N, a) = \int_{x_0}^{1+x_0} W(N, x)\, E(-ax)\, dx$, so that $A(N) = D(N, N)$.

LEMMA 5.1.A. *Suppose that $n \leqslant 2N$. Then*

$$\sum_{\substack{q \leqslant P \\ r|q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} I_2(n, q, a) - J_\beta(N, n)\, G(N, n) \ll NP^{-1} r^2 \phi(r)^{-1} \Sigma_1,$$

*where*

$$\Sigma_1 \ll \log P [\log\log(N+3)]^3\, P^{5/8} \exp(c \log^{3/8} N).$$

*Proof.* By [12, Eq. (9.14)] we have

$$\sum_{\substack{q \leqslant P \\ r \mid q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} I_2(n, q, a) - J_\beta(N, n)\, G(N, n) \ll NP^{-1} r^2 \phi(r)^{-1}\, \Sigma_1,$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{m \mid n \\ (m,r)=1}} \mu(m)^2\, m\phi(m)^{-1} \sum_{\substack{k \leqslant P/rm \\ (k,rn)=1}} \mu(k)^2\, k\phi(k)^{-2} \\ &= \sum_{\substack{m \mid n \\ (m,r)=1 \\ m \leqslant P}} \mu(m)^2\, m\phi(m)^{-1} \sum_{\substack{k \leqslant P/rm \\ (k,rn)=1}} \mu(k)^2\, k\phi(k)^{-2}, \end{aligned}$$

since $k \leqslant P/rm$ has no solution in positive integers $k$ if $m > P$, so that

$$\Sigma_1 \leqslant \sum_{\substack{m \mid n \\ m \leqslant P}} m\phi(m)^{-1} \sum_{k \leqslant P} k\phi(k)^{-2}.$$

The proof now follows from a straightforward application of Lemmas 2.3 and 2.4.

LEMMA 5.2. *Suppose that $n \leqslant 2N$. Then*

$$D(N, n) - J_\beta(N, n)\, G(N, n) \ll NP^{-1} r^2 \phi(r)^{-1}\, \Sigma_1 + N \log^{-2} N r^{-1/2} \log\log^2 N.$$

*Proof.* As in the proof of [12, Lemma 9.8] we have

$$D(N, n) - J_\beta(N, n)\, G(N, n) = A - B,$$

where

$$A = \sum_{\substack{q \leqslant P \\ r \mid q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} I_2(n, q, a) - J_\beta(N, n)\, G(N, n)$$

and

$$B = 2 \sum_{\substack{q \leqslant P \\ r \mid q}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} I_1(n, q, a).$$

By Lemma 5.1A it follows that $A \ll NP^{-1} r^2 \phi(r)^{-1}\, \Sigma_1$, and by [12, Lemma 9.5] we have $B \ll N \log^{-2} N r^{1/2} \phi(r)^{-1}\, n\phi(n)^{-1}$.

The proof now follows from a straightforward application of Lemma 2.4 and [12, Eq. (8.2)].

LEMMA 5.3. *If $n \leqslant 2N$ and $n$ is even, then*

$$|G(N, n)| \leqslant S(n) + O(r^{3/8}P^{-3/8}(\log\log N)^5 \exp(c(\log n)^{3/8})).$$

*Proof.* By Lemma 2.4 and the argument in [12, p. 44]

$$\begin{aligned}
&\sum_{\substack{k \leqslant P/r \\ (k,r)=1}} \frac{\mu(k)^2\, \mu(k/(k,n))}{\phi(k)\, \phi(k/(k,n))} \\
&\quad \leqslant \sum_{\substack{m \mid n \\ m \leqslant P/r \\ (m,r)=1}} \frac{\mu(m)^2}{\phi(m)} \left\{ \left| \sum_{\substack{k \\ (k,nr)=1}} \frac{\mu(k)}{\phi(k)^2} \right| + O(P^{-1} rm (\log\log N)^2) \right\} \\
&\quad \leqslant \sum_{\substack{m \mid n \\ m \leqslant P/r \\ (m,r)=1}} \frac{\mu(m)^2}{\phi(m)} \left| \sum_{\substack{k \\ (k,nr)=1}} \frac{\mu(k)}{\phi(k)^2} \right| \\
&\qquad + O\left( \sum_{\substack{m \mid n \\ m \leqslant P/r \\ (m,r)=1}} \frac{\mu^2(m)}{\phi(m)} P^{-1} rm (\log\log N)^2 \right),
\end{aligned}$$

and by the Euler product formula and Lemma 2.4,

$$\leqslant \sum_{\substack{m \mid n \\ (m,r)=1}} \frac{\mu(m)^2}{\phi(m)} \left| \prod_{p \nmid nr} \left(1 - \frac{1}{(p-1)^2}\right) \right| + O\left( rP^{-1} (\log\log N)^3 \sum_{\substack{m \mid n \\ m \leqslant P/r}} 1 \right),$$

and by the Euler product formula and Lemma 2.3

$$\leqslant \sum_{\substack{p \mid n \\ p \nmid r}} \left(\frac{p}{p-1}\right) \prod_{p \nmid nr} \frac{p(p-2)}{(p-1)^2} + O\left( rP^{-1} (\log\log N)^3 \left(\frac{P}{r}\right)^{5/8} \exp(c \log^{3/8} n) \right).$$

Hence by [12, Eq. (10.1)] it follows that

$$\begin{aligned}
|G(N, n)| &\leqslant \frac{r}{\phi(r)\, \phi\left(\frac{r}{(r,n)}\right)} \prod_{\substack{p \mid n \\ p \nmid r}} \left(\frac{p}{p-1}\right) \prod_{p \nmid nr} \frac{p(p-2)}{(p-1)^2} \\
&\quad + O\left( \frac{r}{\phi(r)\, \phi\left(\frac{r}{(r,n)}\right)} rP^{-1} (\log\log N)^3 \left(\frac{P}{r}\right)^{5/8} \exp(c \log^{3/8} n) \right)
\end{aligned}$$

By a straightforward application of Lemma 2.4

$$\frac{r}{\phi(r)\,\phi(r/(r,n))} \ll (\log\log N)^2;$$

so that the $O$-term it established. The rest of the proof follows from the argument given in [12, p. 45] and the easily established fact that

$$\frac{r}{\phi(r)\,\phi\left(\dfrac{r}{(r,n)}\right)} \prod_{\substack{p\mid n\\ p\nmid r}} \left(\frac{1}{p-1}\right) \prod_{p\nmid nr} \frac{p(p-2)}{(p-1)^2}$$
$$= \left(\prod_{\substack{p\mid r\\ p\nmid n}} \frac{1}{p-1}\right)\left(\prod_{p\mid nr} \frac{p}{p-1}\right)\left(\prod_{p\nmid nr} \frac{p(p-2)}{(p-1)^2}\right)$$

LEMMA 5.4. *Suppose that $n \leqslant 2N$ and $n$ is even. Then*

$$|D(N,n)| \leqslant J_\beta(N,n)\,S(n) + A_2 + A_3 + A_4,$$

*where*

$$A_2 = C_1 N r^{3/8} P^{-3/8} (\log\log N)^5 \exp(c \log^{3/8} N),$$
$$A_3 = C_2 N P^{-1} r (\log\log N)\,\Sigma_1,$$
$$A_4 = C_3 N \log^{-2} N r^{-1/2} \log\log^2 N.$$

*Proof.* By Lemma 5.2

$$D(N,n) = J_\beta(N,n)\,G(N,n) + O(NP^{-1} r^2 \phi^2(r)^{-1}\,\Sigma_1$$
$$+ N \log^{-2} N r^{-1/2} \log\log^2 N).$$

Therefore by Lemma 5.3 and using the fact [12, Lemma 5.5] that $|J_\beta(N,n)| \ll N$, it follows that

$$|D(N,n)| \leqslant J_\beta(N,n)\,S(n) + C_1 N r^{3/8} P^{-3/8} (\log\log N)^5 \exp(c \log^{3/8} N)$$
$$+ C_2 N P^{-1} r^2 \phi(r)^{-1}\,\Sigma_1 + C_3 N \log^{-2} N r^{-1/2} \log\log^2 N.$$

The proof now follows by application of Lemma 2.4.

LEMMA 5.5. *Suppose that $m \leqslant 2N$ and $m$ is even. Then*

$$|J(N,m)\,S(m) + D(N,m)| \geqslant C_{15} r^{-1/8} J(N,m)\,S(m) - A_2 - A_3 - A_4.$$

*Proof.* It is immediate by Lemma 5.4 that

$$|J(N, m)\, S(m) + D(N, m)| \geqslant J(N, m)\, S(m) - J_\beta(N, m)\, S(m) - A_2 - A_3 - A_4.$$

The proof is completed by application of [12, Lemma 10.3].

LEMMA 5.6. *For $\frac{1}{2}N < m \leqslant N$ and $m$ even*

$$|J(N, m)\, S(m) + D(N, m)| \gg NP^{-1/32}(\log N)^{-2}.$$

*Proof.* Let $A_1 = C_{15} r^{-1/8} J(N, m)\, S(m)$. By straightforward calculation it can be shown that $A_4 \leqslant A_1/2$, $A_1 \gg mP^{-1/32} \log^{-2} N$, $A_2 = o(NP^{-1/32} \log^{-2} N)$, and $A_3 = o(NP^{-1/32} \log^{-2} N)$, so that the proof follows immediately from Lemma 5.5.

## 6. AN IMPORTANT COUNTEREXAMPLE

The computer results in [9] indicate that the $(q, N) = 1$ condition in the definition of $m^*(N)$ in Theorem 1.2 might not be a reasonable one. In [10] we indicate that a very natural way to eliminate this condition would require that one show that either $S(N) - R(N) = o(1)$ or $S(N) - R(N) = o(S(N))$.

In this section we show that neither of these conditions is true if $q \leqslant \log^m N$ for any integer $m > 0$. This result follows from the following calculations and lemmas:

Let $E(N) = S(N) - R(N)$. Then

$$E(N) = \sum_{a=1}^{N} \sum_{\substack{(q,N)=a \\ \log^m N < q}} \frac{\mu^2(q)}{\phi^2(q)} C_q(N) = \sum_{a \mid N} \sum_{\substack{(q,N)=a \\ \log^m N < q}} \frac{\mu^2(q)}{\phi^2(q)} C_q(N)$$

$$= \sum_{a \mid N} \sum_{\substack{(q,N)=a \\ \log^m N < q}} \frac{\mu^2(q)}{\phi^2(q)} \frac{\mu(q/a)\, \phi(q)}{\phi(q/a)}$$

by [6, Theorem 272]. Hence

$$E(N) = \sum_{a \mid N} \sum_{\substack{(q/a,N/a)=1 \\ \log^m N < 1}} \frac{\mu^2(q)}{\phi(q)} \frac{\mu(q/a)}{\phi(q/a)}.$$

Now write $q = aa'$, so that

$$E(N) = \sum_{a \mid N} \sum_{\substack{(a',N/a)=1 \\ \log^m N < aa'}} \frac{\mu^2(aa')\, u(a')}{\phi(aa')\, \phi(a')}.$$

But

$$\sum_{\substack{(a', N/a)=1 \\ \log^m N < aa'}} \frac{\mu^2(aa')\, u(a')}{\phi(aa')\, \phi(a')}$$

$$= \sum_{\substack{(a', N/a)=1 \\ \log^m N < aa' \\ (a,a')=1}} \frac{\mu^2(aa')\, u(a')}{\phi(aa')\, \phi(a')} + \sum_{\substack{(a', N/a)=1 \\ \log^m N < aa' \\ (a,a')\neq 1}} \frac{\mu^2(aa')\, u(a')}{\phi(aa')\, \phi(a')}.$$

But if $(a, a') > 1$, then $aa'$ is not square-free, so that $\mu(aa') = 0$. Hence

$$E(N) = \sum_{a \mid N} \sum_{\substack{(a', N/a)=1 \\ \log^m N < aa' \\ (a,a')=1}} \frac{\mu^2(aa')\, u(a')}{\phi(aa')\, \phi(a')} = \sum_{a \mid N} \sum_{\substack{(a', N/a)=1 \\ \log^m N < aa' \\ (a,a')=1}} \frac{u^2(a)\, \mu^3(a')}{\phi(a)\, \phi^2(a')}$$

$$= \sum_{a \mid N} \sum_{\substack{(a', N)=1 \\ \log^m N < aa'}} \frac{\mu^2(a)\, \mu^3(a')}{\phi(a)\, \phi^2(a')}.$$

Now assume $N$ is square-free:

$$E(N) = \sum_{a \mid N} \frac{1}{\phi(a)} \sum_{\substack{(a', N)=1 \\ \log^m N < aa'}} \frac{\mu(a')}{\phi^2(a')}$$

$$= \sum_{\substack{a \mid N \\ a \leqslant \log^m N}} \frac{1}{\phi(a)} \sum_{\substack{(a', N)=1 \\ \log^m N < aa'}} \frac{\mu(a')}{\phi^2(a')}$$

$$+ \sum_{\substack{a \mid N \\ a > \log^m N}} \frac{1}{\phi(a)} \sum_{\substack{(a', N)=1 \\ \log^m N < aa'}} \frac{\mu(a')}{\phi^2(a')} = A(N) + B(N).$$

$$B(N) = \sum_{\substack{a \mid N \\ a > \log^m N}} \frac{1}{\phi(a)} \sum_{\substack{(a', N)=1 \\ 1 \leqslant a'}} \frac{\mu(a')}{\phi^2(a')}$$

$$= \left[ \sum_{\substack{(a', N)=1 \\ 1 \leqslant a'}} \frac{\mu(a')}{\phi^2(a')} \right] \cdot \left[ \sum_{\substack{a \mid N \\ a > \log^m N}} \frac{1}{\phi(a)} \right]$$

$$= B'(N) \cdot B''(N).$$

LEMMA 6.1. *We have* $B'(N) \gg 1$.

*Proof.* We have

$$B'(N) = \sum_{\substack{(a',N)=1 \\ 1 \leqslant a'}} \frac{\mu(a')}{\phi^2(a')} = \prod_{p \nmid N} \left(1 - \frac{1}{(p-1)^2}\right)$$

$$\geqslant \prod_{p} \left(1 - \frac{1}{(p-1)^2}\right) \geqslant \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^2}\right) \geqslant C.$$

LEMMA 6.2. *For each $n$ let $Q(n) = p_1 \cdot p_2 \cdots p_n$. For each $V$ such that $C_0 \leqslant V \leqslant \log^{2m+3} Q(n)$ the number of square-free integers which are less than or equal to $V$ and which divide $Q(n)$ is at least $cV$.*

*Proof.* This follows from a suitable modification of the proof of [1, Lemma on p. 8].

LEMMA 6.3. *For each $n$ let $Q(n) = p_1 \cdot p_2 \cdots p_n$. There exists a $D > 1$ such that for each $W$ such that $C_0 < W \leqslant \log^{2m} Q(n)$*

$$\sum_{\substack{a \mid Q(n) \\ a > W \\ a \leqslant DW \\ a \text{ square-free}}} \frac{1}{a} \gg 1.$$

*Proof.* We have

$$\sum_{\substack{a \mid Q(n) \\ a > W \\ a \leqslant DW \\ a \text{ square-free}}} \frac{1}{a} \geqslant \frac{1}{DW} \sum_{\substack{a \mid Q(n) \\ a > W \\ a \leqslant DW \\ a \text{ square-free}}} 1$$

$$\geqslant \frac{1}{DW} \Bigg[ \sum_{\substack{a \mid Q(n) \\ a \leqslant DW \\ a \text{ square-free}}} 1 - \sum_{\substack{a \mid Q(n) \\ a \leqslant W \\ a \text{ square-free}}} 1 \Bigg].$$

The proof now follows immediately from Lemma 6.2.

LEMMA 6.4. *We have $B''(Q(n)) \gg \log\log Q(n)$.*

*Proof.* We can break $B''(Q(n))$ into $\gg \log\log Q(n)$ sums of the form in Lemma 6.3.

LEMMA 6.5. *We have $S(Q(n)) \ll \log\log Q(n)$.*

*Proof.* By straightforward calculation it follows that

$$S(Q(n)) \ll \prod_{\substack{p \leqslant p_n \\ p \geqslant 3}} \left(1 - \frac{1}{p}\right)^{-1},$$

so that the proof follows by an application of [6, Theorem 429].

LEMMA 6.6. *We have*

$$\sum_{y \leqslant N} \frac{1}{\phi(N)^2} \ll \frac{1}{y}.$$

*Proof.* This is immediate by Lemma 6.10.

LEMMA 6.7 *We have* $A(N) \ll 1$.

*Proof.* This follows from Lemma 6.9 and Lemma 6.11:

$$\begin{aligned} |A(N)| &\leqslant \sum_{\substack{a \mid N \\ a \leqslant \log^m N}} \frac{1}{\phi(a)} \sum_{\substack{(a',N)=1 \\ \log^m N/a < a'}} \frac{1}{\phi(a')^2} \\ &\leqslant \sum_{a \leqslant \log^m N} \frac{1}{\phi(a)} \sum_{\log^m N/a < a'} \frac{1}{\phi(a')^2} \\ &\ll \frac{1}{\log^m N} \sum_{a \leqslant \log^m N} \frac{a}{\phi(a)} \ll \left(\frac{1}{\log^m N}\right) (\log^m N) = 1. \end{aligned}$$

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